JAN 12

1. Differentiate with respect to $x$, giving your answer in its simplest form,

(a) $x^2 \ln(3x)$

(b) $\dfrac{\sin 4x}{x^3}$

(5)

a) $y = uv \quad \dfrac{dy}{dx} = vu' + uv'$

$u = x^2 \quad v = \ln(3x)$

$u' = 2x \quad v' = \dfrac{3}{3x} = \dfrac{1}{x}$

$\therefore \dfrac{dy}{dx} = 2x\ln(3x) + x$

b) $y = \dfrac{u}{v} \quad \dfrac{dy}{dx} = \dfrac{vu' - uv'}{v^2}$

$u = \sin 4x \quad v = x^3$

$u' = 4\cos 4x \quad v' = 3x^2$

$\therefore \dfrac{dy}{dx} = \dfrac{4x^3\cos 4x - 3x^2\sin 4x}{x^6} = \dfrac{4x\cos 4x - 3\sin 4x}{x^4}$

**2.**

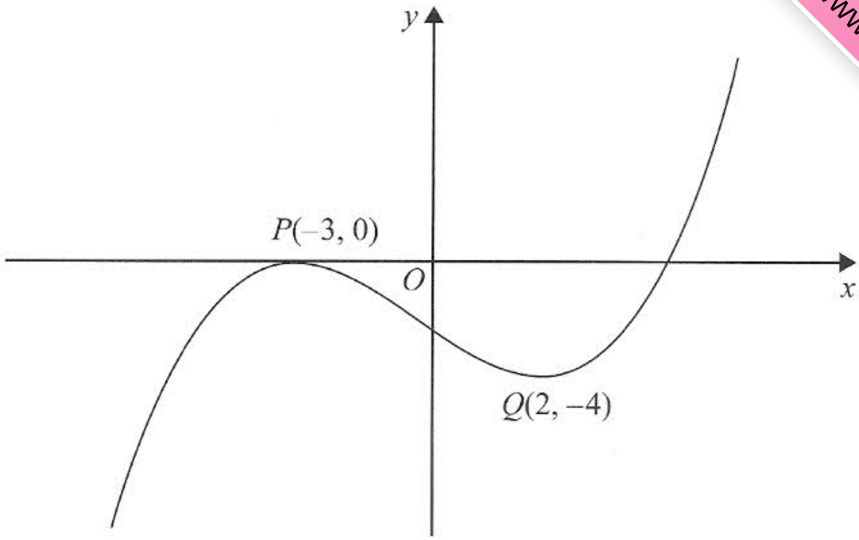



**Figure 1**

Figure 1 shows the graph of equation $y = \mathrm{f}(x)$.

The points $P(-3, 0)$ and $Q(2, -4)$ are stationary points on the graph.

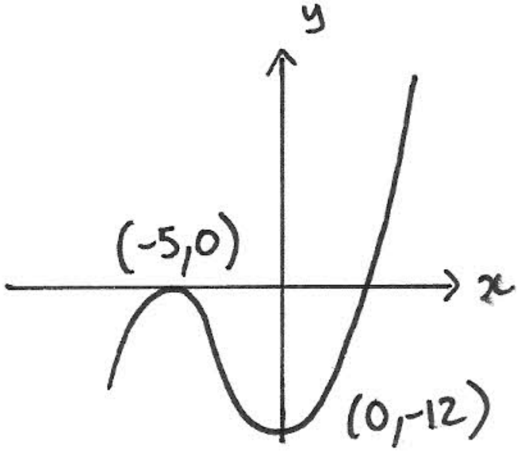
Sketch, on separate diagrams, the graphs of

(a) $y = 3\mathrm{f}(x+2)$ $\quad$ ↕ $y \times 3$ $\quad$ $\xleftarrow[2]{}$ $x - 2$ **(3)**

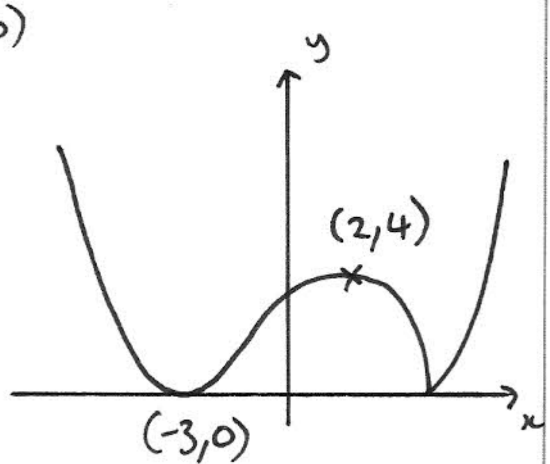
(b) $y = |\mathrm{f}(x)|$ $\quad$ bounce $x$ **(3)**

On each diagram, show the coordinates of any stationary points.

a)




b)



3. The area, $A$ mm$^2$, of a bacterial culture growing in milk, $t$ hours after ... is gi...

$$A = 20e^{1.5t}, \quad t \geqslant 0$$

(a) Write down the area of the culture at midday.

$t = 0 \Rightarrow \underline{A = 20}$

(b) Find the time at which the area of the culture is twice its area at midday. Give your answer to the nearest minute.

(5)

b) $2 \times \cancel{20} = \cancel{20}e^{1.5t} \Rightarrow \ln 2 = 1.5t \Rightarrow t = 0.462\ldots$

$\therefore \underline{12:28}$

**4.** The point $P$ is the point on the curve $x = 2\tan\left(y + \frac{\pi}{12}\right)$ with $y$-coordi...

Find an equation of the normal to the curve at $P$.

(7)



$x_1 = 2\tan\left(\frac{\pi}{4} + \frac{\pi}{12}\right) = 2\tan\left(\frac{\pi}{3}\right) = 2\sqrt{3}$

$\frac{dx}{dy} = 2\sec^2\left(y + \frac{\pi}{12}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{2}\cos^2\left(y + \frac{\pi}{12}\right)$

at $P$ $M_t = \frac{1}{2}\cos^2\left(\frac{\pi}{4} + \frac{\pi}{12}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi}{3}\right)\right]^2 = \frac{1}{2} \times \left(\frac{1}{2}\right)^2$

$= \frac{1}{8}$

$\therefore M_n = -8$

$\therefore y - y_1 = m(x - x_1) \Rightarrow y - \frac{\pi}{4} = -8(x - 2\sqrt{3})$

5. Solve, for $0 \leqslant \theta \leqslant 180°$,

$$2\cot^2 3\theta = 7\operatorname{cosec} 3\theta - 5$$

Give your answers in degrees to 1 decimal place.

$$\frac{\sin^2}{\sin^2} + \frac{\cos^2}{\sin^2} = \frac{1}{\sin^2} \Rightarrow 1 + \cot^2 = \operatorname{cosec}^2$$

$$\Rightarrow \cot^2 = \operatorname{cosec}^2 - 1$$

$$\Rightarrow 2\operatorname{cosec}^2 3\theta - 2 = 7\operatorname{cosec} 3\theta - 5$$

$$\Rightarrow 2\operatorname{cosec}^2 3\theta - 7\operatorname{cosec} 3\theta + 3 = 0$$

$$\Rightarrow (2\operatorname{cosec} 3\theta - 1)(\operatorname{cosec} 3\theta - 3) = 0$$

$\operatorname{cosec} 3\theta = \frac{1}{2}$ $\qquad$ $\operatorname{cosec} 3\theta = 3$

$\sin 3\theta = 2$ $\qquad$ $\sin 3\theta = \frac{1}{3}$

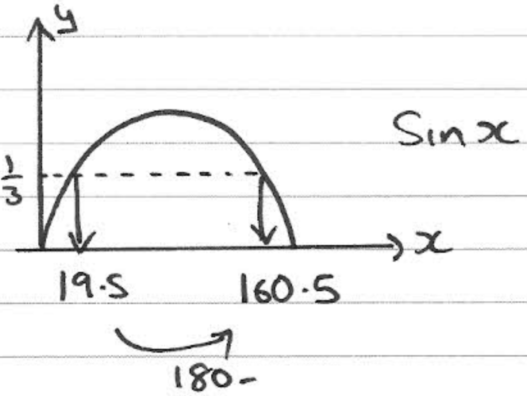
no solutions! $\qquad$ $\Rightarrow 3\theta = \sin^{-1}\left(\frac{1}{3}\right)$

180 − $\qquad$ +360

$$\Rightarrow 3\theta = 19.47..., 160.53..., 379.47..., 520.53$$

+360

$$\therefore \theta = 6.5°, 53.5°, 126.5°, 173.5°$$



6. $$f(x) = x^2 - 3x + 2\cos\left(\tfrac{1}{2}x\right), \quad 0 \leqslant x \leqslant \pi$$

(a) Show that the equation $f(x) = 0$ has a solution in the interval $0.8 < x < 0.$ **(2)**



The curve with equation $y = f(x)$ has a minimum point $P$.

(b) Show that the $x$-coordinate of $P$ is the solution of the equation

$$x = \frac{3 + \sin\left(\frac{1}{2}x\right)}{2}$$

**(4)**

(c) Using the iteration formula

$$x_{n+1} = \frac{3 + \sin\left(\frac{1}{2}x_n\right)}{2}, \quad x_0 = 2$$

find the values of $x_1$, $x_2$ and $x_3$, giving your answers to 3 decimal places.

**(3)**

(d) By choosing a suitable interval, show that the $x$-coordinate of $P$ is 1.9078 correct to 4 decimal places.

**(3)**

a) $f(0.8) = 0.082$, $f(0.9) = -0.089$ } change of sign $\Rightarrow$ root lies between 0.8 and 0.9

b) min point $\Rightarrow f'(x) = 0 \quad \therefore 2x - 3 - \sin\left(\frac{1}{2}x\right) = 0$

$\Rightarrow 2x = 3 + \sin\left(\frac{1}{2}x\right) \Rightarrow x = \dfrac{3 + \sin\left(\frac{1}{2}x\right)}{2}$

c) $x_0 = 2$ ; $x_1 = 1.921$ ; $x_2 = 1.910$ ; $x_3 = 1.908$

d) $f'(1.90775) = -0.00016$

$f'(1.90785) = 0.0000077$

change of sign $\Rightarrow f'(x) = 0$ when $x$ is between 1.90775 and 1.90785

$\therefore x = 1.9078$ (4dp)

7. The function f is defined by

$$f : x \mapsto \frac{3(x+1)}{2x^2+7x-4} - \frac{1}{x+4}, \quad x \in \mathbb{R}, x > \frac{1}{2}$$

(a) Show that $f(x) = \frac{1}{2x-1}$ **(4)**

(b) Find $f^{-1}(x)$ **(3)**

(c) Find the domain of $f^{-1}$ **(1)**

$$g(x) = \ln(x+1)$$

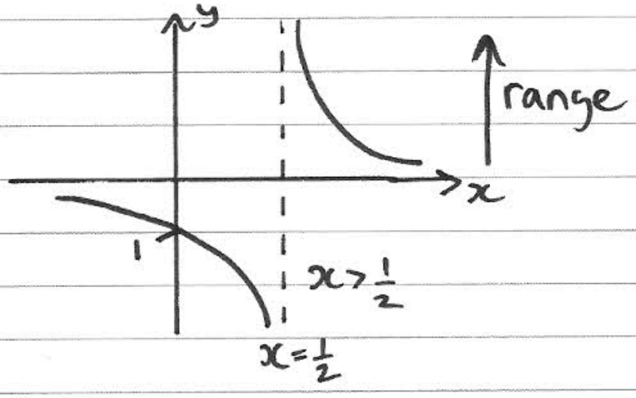
(d) Find the solution of $fg(x) = \frac{1}{7}$, giving your answer in terms of e. **(4)**

a) $\frac{3(x+1)}{(2x-1)(x+4)} - \frac{1}{(x+4)} \frac{\times(2x-1)}{\times(2x-1)} = \frac{3x+3-2x+1}{(2x-1)(x+4)}$

$= \frac{x+4}{(2x-1)(x+4)} = \frac{1}{2x-1}$ #

b) $x = \frac{1}{2y-1} \Rightarrow 2y-1 = \frac{1}{x} \Rightarrow 2y = \frac{1}{x}+1 \Rightarrow y = \frac{1}{2x} + \frac{1}{2}$ $f^{-1}(x) =$

c) domain of $f'(x)$ = range of $f(x)$




$y = \frac{1}{2x-1}$ $\quad x \neq \frac{1}{2}$ asymptote

$x = 0 \Rightarrow y = -1 \quad (0,-1)$

$\therefore$ range of $f(x)$ $y > 0$

$\therefore$ domain of $f^{-1}(x)$ $x > 0$

d) $fg(x) = f[\ln(x+1)] = \dfrac{1}{2\ln(x+1)-1} = 7$

$\therefore\ 2\ln(x+1) - 1 = 7 \Rightarrow 2\ln(x+1) = 8 \Rightarrow \ln(x+1$

$\Rightarrow x+1 = e^4 \Rightarrow x = \underline{-1+e^4}$

8. (a) Starting from the formulae for $\sin(A+B)$ and $\cos(A+B)$, prove tha

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(b) Deduce that

$$\tan\left(\theta + \frac{\pi}{6}\right) = \frac{1+\sqrt{3}\tan\theta}{\sqrt{3}-\tan\theta}$$

**(3)**

(c) Hence, or otherwise, solve, for $0 \leqslant \theta \leqslant \pi$,

$$1+\sqrt{3}\tan\theta = (\sqrt{3}-\tan\theta)\tan(\pi-\theta)$$

Give your answers as multiples of $\pi$.

**(6)**



a) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

$= \dfrac{\frac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} + \frac{\cancel{\cos A}\sin B}{\cancel{\cos A}\cos B}}{\frac{\cancel{\cos A}\cancel{\cos B}}{\cancel{\cos A}\cancel{\cos B}} - \frac{\sin A \sin B}{\cos A \cos B}} \Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B}$ #

b) $\tan\left(\theta + \frac{\pi}{6}\right) = \frac{\tan\theta + \tan\left(\frac{\pi}{6}\right)}{1 - \tan\theta\tan\left(\frac{\pi}{6}\right)} = \frac{\tan\theta + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}\tan\theta}$ $\times\sqrt{3}$ $\times\sqrt{3}$

$= \frac{\sqrt{3}\tan\theta + 1}{\sqrt{3} - \tan\theta}$ #

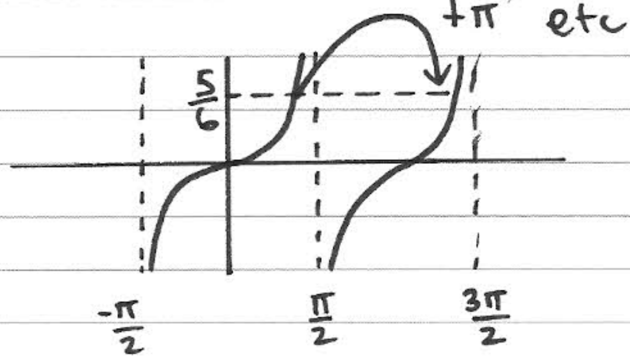
c) $1+\sqrt{3}\tan\theta = (\sqrt{3}-\tan\theta)\tan(\pi-\theta)$

$\Rightarrow \tan\left(\theta+\frac{\pi}{6}\right) = \frac{\cancel{(\sqrt{3}-\tan\theta)}\tan(\pi-\theta)}{\cancel{\sqrt{3}-\tan\theta}}$

$\therefore \tan\left(\theta + \frac{\pi}{6}\right) = \tan(\pi - \theta) \Rightarrow \theta + \frac{\pi}{6} = \pi - \theta$

$\Rightarrow 2\theta = \frac{5}{6}\pi$

Question 8 continued




$2\theta = \frac{5}{6}\pi, \frac{11\pi}{6},$

$\theta = \frac{5}{12}\pi, \frac{11\pi}{12}$

$\tan\pi = 0$

alt

$\tan(\pi-\theta) = \frac{\tan\pi - \tan\theta}{1+\tan(\pi)\tan\theta} = \frac{-\tan\theta}{1} = -\tan\theta$

$\Rightarrow 1+\sqrt{3}\tan\theta = (\sqrt{3}-\tan\theta)\times -\tan\theta$

$\Rightarrow 1+\sqrt{3}\tan\theta = -\sqrt{3}\tan\theta + \tan^2\theta$

$\Rightarrow \tan^2\theta - 2\sqrt{3}\tan\theta - 1 = 0$

$\Rightarrow \tan\theta = \frac{2\sqrt{3} \pm \sqrt{(2\sqrt{3})^2 - 4(-1)}}{2} = \frac{2\sqrt{3}\pm 4}{2} = \sqrt{3}\pm 2$

$\theta = \tan^{-1}(\sqrt{3}+2) \Rightarrow \theta = \frac{5\pi}{12}$

$\theta = \tan^{-1}(\sqrt{3}-2) \Rightarrow \theta = -\frac{1}{12}\pi \xrightarrow{+\pi} \frac{11}{12}\pi$

$\therefore \theta = \frac{5\pi}{12}, \frac{11\pi}{12}$